Slow flow past ellipsoids of revolution

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The results of Proudman & Pearson (1957) for a sphere are generalized to apply to all ellipsoids of revolution both prolate and oblate.

1. Introduction

The two classical methods, those of Stokes (1851) and Oseen (1910), for finding approximations to viscous streaming at low Reynolds numbers have both been applied to the problem of determining the flow past an ellipsoid in a uniform stream. By using the first method Oberbeck (1876) obtained a first approximation for any ellipsoid at any orientation to a uniform stream. For the particular cases of bodies with rotational symmetry about an axis parallel to the stream the results can be expressed in terms of a Stokes stream function, ψ. Thus, for a prolate spheroid of eccentricity e and with semi-major axis of unit length in a stream of velocity U, Oberbeck's result becomes

$$\psi = \tfrac{1}{2}U\left\{(r^2-e^2) - \frac{\left[e(1-e^2)r - \tfrac{1}{2}(1+e^2)(r^2-e^2)\log\left(\dfrac{r+e}{r-e}\right)\right]}{\left[e - \tfrac{1}{2}(1+e^2)\log\left(\dfrac{r+e}{r-e}\right)\right]}\right\}\sin^2\theta, \quad (1.1)$$

where (r, θ, ϕ) are prolate spheroidal coordinates. Oseen's method was applied by him to many similar cases the results for which are collected in his book of 1927. This paper gives the results of attempting to find higher approximations for slow flow past ellipsoids of revolution by a technique which combines Stokes's and Oseen's methods. This technique has been used by Proudman & Pearson (1957) and Kaplun & Lagerstrom (1957) to obtain higher approximations to the flow past a sphere and circular cylinder.

The Stokes approximation breaks down in the outer region where the neglected inertia terms are comparable in magnitude with the retained viscous terms. On moving away from the body this region is reached when $Rr = O(1)$, where R is the Reynolds number. In the technique mentioned above the co-ordinate system is then strained by taking new variables $\rho = Rr$ and $\Psi(\rho, \cos\theta) = R^2\psi(\rho, \cos\theta)$. Ψ is chosen so that there are no first-order terms in R in the transformed equations of motion. Expansions of the forms

$$\psi = \sum_{n=0}^{\infty} f_n(R)\,\psi_n(r, \cos\theta) \quad \text{and} \quad \Psi = \sum_{n=0}^{\infty} F_n(R)\,\Psi_n(\rho, \cos\theta), \quad (1.2)$$

where
$$\frac{f_{n+1}(R)}{f_n(R)} \to 0 \quad \text{and} \quad \frac{F_{n+1}(R)}{F_n(R)} \to 0 \quad \text{as} \quad R \to 0, \quad (1.3)$$

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and F_0 and f_0 can be taken as bounded for small R, are assumed for ψ and Ψ. These are called the Stokes and Oseen expansions, respectively, after the leading terms in each. The no-slip condition on the body partially determines the ψ_n. The condition that the solution must tend to the uniform stream far out partially determines the Ψ_n. The ψ_n and Ψ_n are completely determined by matching the expansions (1.2) in orders of R as ρ becomes small and r becomes large. For this matching to be possible there must exist a region where the Oseen expansion and the Stokes expansion overlap. The existence of such a region is discussed by Kaplun & Lagerstrom (1957).

For a prolate spheroid with its axis along the uniform stream the following results in terms of non-dimensional variables have been found

$$\Psi = \tfrac{1}{2}\rho^2(1-\mu^2) - R\frac{B}{6}\{1-\exp[-\tfrac{1}{2}\rho(1-\mu)]\}(1+\mu) + O(R^2) \tag{1.4}$$

and
$$\psi = \left\{\tfrac{1}{2} + R\frac{B}{48} + R^2\log R\frac{B^2}{720}\right\}\left\{(r^2-e^2) - \frac{Br}{6} + \frac{B(1+e^2)}{6e}V_1(r/e)\right\}(1-\mu^2)$$
$$+R\sum_{m=1}^{\infty} X_{2m}(r,e)\,U_{2m}(\mu) + O(R^2), \tag{1.5}$$

where
$$B = 24e^3\left\{(1+e^2)\log\left(\frac{1+e}{1-e}\right) - 2e\right\}^{-1}, \tag{1.6}$$

$\mu = \cos\theta$ and the Reynolds number, R, is Ua/ν, where ν is the kinematic viscosity and a is the semi-major axis of the ellipsoid. The functions $U_n(x)$ and $V_n(x)$ are defined by

$$U_n(x) = \int_1^x P_n(y)\,dy \quad \text{and} \quad V_n(x) = \int_x^\infty Q_n(y)\,dy, \tag{1.7}$$

where $P_n(y)$ and $Q_n(y)$ are Legendre's functions of the first and second kind of the nth degree. X_{2m} is a complicated function of r and e which is $O(r^{-2m+4})$ for large r. Its structure is given more explicitly below in §4.

2. The equations of motion and the Oseen terms

Prolate spheroidal coordinates are related to rectangular cartesian coordinates by the equations

$$\left.\begin{aligned} x &= \sqrt{(r^2-d^2)}\sin\theta\cos\phi,\\ y &= \sqrt{(r^2-d^2)}\sin\theta\sin\phi,\\ z &= r\cos\theta, \end{aligned}\right\} \tag{2.1}$$

where r is the semi-major axis of the prolate spheroid which passes through (x, y, z) and θ is the eccentric angle of that point with respect to the ellipse in which a plane through the axis of symmetry cuts the spheroid. ϕ corresponds to the azimuthal angle of spherical polars. The spheroids are confocal with foci $(0, 0, \pm d)$.

If the flow under consideration is that past a stationary prolate spheroid of semi-major axis a and eccentricity e with its centre at the origin and its axis of symmetry parallel to an otherwise undisturbed uniform stream of velocity U, then non-dimensional (primed) variables can be defined by

$$r = ar', \quad \psi = a^2U\psi', \quad \mu = \mu'. \tag{2.2}$$

Further, if the foci of the ellipsoid coincide with the foci of the coordinate system then $d = ae$. The Navier–Stokes equations for incompressible, steady, viscous flow then yield, on dropping the primes and introducing a new dependent variable l,

$$E_r^2\psi = -(r^2-e^2)(r^2-e^2\mu^2)(1-\mu^2)\,l \tag{2.3}$$

and

$$\bar{E}_r^2 l = R\frac{\partial(\psi, l)}{\partial(r,\mu)}, \tag{2.4}$$

where

$$E_r^2 \equiv (r^2-e^2)\frac{\partial^2}{\partial r^2}+(1-\mu^2)\frac{\partial^2}{\partial \mu^2} \tag{2.5}$$

and $\bar{E}_r^2$ is the adjoint operator

$$(r^2-e^2)\frac{\partial^2}{\partial r^2}+(1-\mu^2)\frac{\partial^2}{\partial \mu^2}+4r\frac{\partial}{\partial r}-4\mu\frac{\partial}{\partial \mu}. \tag{2.6}$$

The variables in these equations are called Stokes variables. The dependent variable l which has been introduced for convenience in solving the equations of motion can be identified physically as the vorticity at a point divided by the distance of that point from the axis of symmetry and may thus be called ring vorticity strength. It will have an expansion of the form

$$l = \sum_{n=0}^{\infty} f_n(R)\, l_n(r,\mu). \tag{2.7}$$

If the Oseen variables, ρ, L, Ψ given by

$$\rho = Rr, \quad \Psi = R^2\psi, \quad L = l \tag{2.8}$$

are used then the equations of motion become

$$(\rho^2-R^2e^2)\frac{\partial^2\Psi}{\partial\rho^2}+(1-\mu^2)\frac{\partial^2\Psi}{\partial\mu^2} = -(\rho^2-R^2e^2)(\rho^2-R^2e^2\mu^2)(1-\mu^2)R^{-2}L \tag{2.9}$$

and

$$(\rho^2-R^2e^2)\frac{\partial^2L}{\partial\rho^2}+4\rho\frac{\partial L}{\partial\rho}+(1-\mu^2)\frac{\partial^2L}{\partial\mu^2}-4\mu\frac{\partial L}{\partial\mu} = \frac{\partial(\Psi, L)}{\partial(\rho,\mu)}. \tag{2.10}$$

By noting that terms in R^2e^2 in (2.9) and (2.10) may be neglected if a solution correct to $O(R)$ in R is sought, the analysis for the first two Oseen terms reduces to that for the sphere case. The constant B appears in (1.4) as a result of matching with the Stokes expansion. The Oseen terms will therefore not be further discussed.

3. The leading Stokes terms

If $f_0(R) = 1$ then from (1.2), (2.3) and (2.4) an appropriate general solution for l_0 finite on $\mu = \pm 1$ and bounded for small R at large values of r is

$$l_0 = \sum_{n=1}^{\infty} \frac{B_n V_n(\tau)\, U_n(\mu)}{e^4(\tau^2-1)(1-\mu^2)}, \tag{3.1}$$

where $\tau = re^{-1}$ and the B_n are constants. By use of the recurrence relationship

$$(2n+1)\mu U_n(\mu) = (n+2)\,U_{n+1}(\mu)+(n-1)\,U_{n-1}(\mu) \tag{3.2}$$

and the variation-of-parameters technique this leads to

$$\psi_0 = \sum_{n=1}^{\infty} \left\{ U_n(\tau) \left[\frac{B_{n+2} v_n^2(\tau)}{(2n+5)(2n+3)} - \frac{2B_n v_n^0(\tau)}{(2n+3)(2n-1)} + \frac{B_{n-2} v_n^{-2}(\tau)}{(2n-1)(2n-3)} - B_n \int_{e^{-1}}^{\tau} \frac{[V_n(x)]^2}{n(n+1)} dx \right] - V_n(\tau) \left[\frac{B_{n+2} u_n^2(\tau)}{(2n+5)(2n+3)} - \frac{2B_n u_n^0(\tau)}{(2n+3)(2n-1)} + \frac{B_{n-2} u_n^{-2}(\tau)}{(2n-1)(2n-3)} - B_n \int_{e^{-1}}^{\tau} \frac{U_n(x) V_n(x)}{n(n+1)} dx \right] \right\} n^2(n+1)^2 U_n(\mu), \quad (3.3)$$

where

$$v_n^m(\tau) = \int_{e^{-1}}^{\tau} \frac{V_n(x) V_m(x)}{(x^2-1)} dx,$$

$$u_n^m(\tau) = \int_{e^{-1}}^{\tau} \frac{U_n(x) V_m(x)}{(x^2-1)} dx$$

and $B_0 = B_{-1} = B_{-2} = 0$.

This satisfies the boundary conditions $\psi = \psi_r = 0$ on $r = 1$. Because of the behaviour of $V_n(\tau)$ at infinity the integrals $v_n^m(\tau)$, $u_n^2(\tau)$ and $\int_{e^{-1}}^{\tau} [V_n(x)]^2 dx$ are convergent as $\tau \to \infty$. The integrals u_n^0, u_n^{-2} and $\int_{e^{-1}}^{\tau} U_n(x) V_n(x) dx$ are of orders $\log \tau$, τ^2 and τ^2, respectively. However, in (3.3) they are multiplied by $V_n(\tau)$ which is $O(\tau^{-n})$ and therefore far out the product of $V_n(\tau)$ and its cofactor in (3.3) is $O(\tau^{-2n+4})$.

When the integrals in the cofactor of $U_n(\tau)$ in (3.3) are evaluated at their upper limits the result is a function which is $O(\tau^{-2n+1})$ for large τ. When this is multiplied by $U_n(\tau)$, which is $O(\tau^{n+1})$, the resulting contribution to ψ_0 is a term $O(\tau^{-n+2})$ and this is not large enough to affect the matching process. However, when the integrals in the cofactor of $U_n(\tau)$ are evaluated at their lower limits the result is a constant. This implies that if ψ_0 is expressed in Oseen variables then the function $U_n(\tau)$ with its cofactor will contribute a term $O(R^{-n+1})$ to Ψ. But $\Psi = O(1)$ for small R. Therefore the constant must be zero for $n > 1$. Hence

$$\frac{B_{n+2} v_n^2(\infty)}{(2n+5)(2n+3)} - \frac{2B_n v_n^0(\infty)}{(2n+3)(2n-1)} + \frac{B_{n-2} v_n^{-2}(\infty)}{(2n-1)(2n-3)} - B_n \int_{e^{-1}}^{\infty} \frac{[V_n(x)]^2}{n(n+1)} dx = 0 \quad (n \geqslant 2), \quad (3.4)$$

since all the integrals vanish at the upper limit if that limit is infinite.

Clearly the infinite set of linear equations (3.4) can be divided into two independent subsets, one involving all the odd B_n's only and the other involving all the even B_n's only. The set for the odd B_n's is not complete because the contribution of Ψ_0 to ψ, namely $\frac{1}{2}r^2(1-\mu^2)$, has not yet been matched to $O(1)$ in R with ψ_0. When this is done a further equation for B_1 and B_3 is obtained. This is

$$\int_{e^{-1}}^{\infty} \{(5x^2-1) V_1(x) B_1 - \tfrac{2}{7} V_3(x) B_3\} \frac{V_1(x)}{(x^2-1)} dx = 5e^2. \quad (3.5)$$

When the help of the recurrence relationship (3.2), which is true in general for $V_n(\tau)$, the case $n = 1$ being exceptional, it can now be shown that for the odd B_n's

$$B_{2m+1} = (2m+2)(2m+1)(4m+3)\left\{\frac{B_1}{6} - M \sum_{s=0}^{m-1} \frac{(2s)!\,(4s+5)}{(2s+4)!\,v_{2m+1}^2(\infty)}\right\},$$

where
$$M = \frac{B_1}{5}\int_{e^{-1}}^{\infty} \{(5x^2-1)V_1(x) - 4V_3(x)\}\frac{V_1(x)}{(x^2-1)}\,dx - e^2.$$

An investigation of the convergence of the odd terms of l_0 shows that the only solution which remains bounded in the field of the flow is that for which $M = 0$. It then follows that

$$B_1 = eB \quad \text{and} \quad B_{2m+1} = (2m+2)(2m+1)(4m+3)\frac{B_1}{6}. \tag{3.6}$$

A similar investigation for the even B_n's shows that the only possible solution to fit the physical conditions is

$$B_{2m} = 0. \tag{3.7}$$

The cofactor of $U_n(\tau)$ in (3.3) then vanishes identically for $n \geqslant 2$ by (3.6), (3.7) and (3.2). The same is true of the cofactor of $V_n(\tau)$ ($n \geqslant 2$) and by the special case of the recurrence relationship for $V_n(\tau)$ when $n = 1$ it is found that

$$\psi_0 = \frac{1}{2}\left\{(r^2-e^2) - \frac{Br}{6} + \frac{B}{6e}(1+e^2)V_1(r/e)\right\}(1-\mu^2), \tag{3.8}$$

which, since

$$V_1(\tau) = \frac{1}{2}\left\{\tau - \tfrac{1}{2}(\tau^2-1)\log\left(\frac{\tau+1}{\tau-1}\right)\right\},$$

agrees with Oberbeck's result (1.1). Further, as $e \to 0$, $B \to 9$ and $e^{-1}V_1(\tau) \to \frac{1}{3}r$, so when the eccentricity is zero and the body is a sphere (3.8) simplifies to the classical Stokes result,

$$\psi_0 = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)(1-\mu^2). \tag{3.9}$$

4. The second Stokes terms

If it assumed that $f_1(R) = R$ then the equation for l_1 is

$$\bar{E}_r^2 l_1 = \frac{\partial(\psi_0, l_0)}{\partial(r,\mu)}. \tag{4.1}$$

If $\psi_0 = T(\tau)(1-\mu^2)$ then the Jacobian in (4.1) can be written

$$\frac{B}{6e^2}\sum_{m=1}^{\infty} 2m(2m+1)(4m+1)\frac{W_{2m}(\tau)}{(\tau^2-1)}\frac{U_{2m}(\mu)}{(1-\mu^2)},$$

where
$$W_{2m}(\tau) = 2(\tau^2-1)V_{2m}(\tau)\frac{d}{d\tau}\left\{\frac{\tau T(\tau)}{(\tau^2-1)}\right\} + V'_{2m}(\tau)\frac{d}{d\tau}\{(\tau^2-1)T(\tau)\}. \tag{4.2}$$

The appropriate general solution for l_1 is then found by variation of parameters to be

$$l_1 = \sum_{n=1}^{\infty}\{C_n V_n(\tau) + Y_n(\tau)\}\frac{U_n(\mu)}{e^4(\tau^2-1)(1-\mu^2)}, \tag{4.3}$$

where C_n is a constant and $Y_n(\tau)$ is such that

$$Y_{2m}(\tau) = \frac{B}{6}(2m)^2(2m+1)^2(4m+1)\left\{U_{2m}(\tau)\int_{\infty}^{\tau}\frac{W_{2m}(x)V_{2m}(x)}{(x^2-1)}dx - V_{2m}(\tau)\int_{e^{-1}}^{\tau}\frac{W_{2m}(x)U_{2m}(x)}{(x^2-1)}dx\right\}$$

and
$$Y_{2m+1}(\tau) = 0. \tag{4.4}$$

For large τ, $Y_{2m}(\tau) = O(\tau^{-2m+2})$.

The general solution for ψ_1 satisfying the boundary conditions on the body is then

$$\psi_1 = \sum_{n=1}^{\infty}\left\{U_n(\tau)\left[\frac{C_{n+2}v_n^2(\tau)+y_n^2(\tau)}{(2n+5)(2n+3)} - \frac{2C_nv_n^0(\tau)+2y_n^0(\tau)}{(2n+3)(2n-1)} + \frac{C_{n-2}v_n^{-2}(\tau)+y_n^{-2}(\tau)}{(2n-1)(2n-3)} - C_n\int_{e^{-1}}^{\tau}\frac{[V_n(x)]^2}{n(n+1)}dx\right]\right.$$
$$\left. - V_n(\tau)\left[\frac{C_{n+2}u_n^2(\tau)+z_n^2(\tau)}{(2n+5)(2n+3)} - \frac{2C_nu_n^0(\tau)+2z_n^0(\tau)}{(2n+3)(2n+1)} + \frac{C_{n-2}u_n^{-2}(\tau)+z_n^{-2}(\tau)}{(2n-1)(2n-3)} - C_n\int_{e^{-1}}^{\tau}\frac{U_n(x)V_n(x)}{n(n+1)}dx\right]\right\}n^2(n+1)^2U_n(\mu), \tag{4.5}$$

where
$$y_n^m(\tau) = \int_{e^{-1}}^{\tau}\frac{V_n(x)Y_m(x)}{(x^2-1)}dx,$$
$$z_n^m(\tau) = \int_{e^{-1}}^{\tau}\frac{U_n(x)Y_m(x)}{(x^2-1)}dx$$

and $C_0 = C_{-1} = C_{-2} = 0$.

As in ψ_0 the constant arising from the evaluation of the integrals in the cofactor of $U_n(\tau)$ in (4.5) at their lower limits must be zero for otherwise ψ_1 would contain terms $O(\tau^{2m+1})$ and would therefore contribute terms $O(R^{-2m+1})$ for small R to Ψ. This again leads to two infinite sets of linear equations for the constants C_n, one for the odd C_n's and one for the even ones. For the latter the only solution for which l_0 remains bounded near the body is given by

$$C_{2m+2} - \frac{(2m+3)(2m+2)(4m+5)}{(2m+1)(2m)(4m+1)}C_{2m} = -y_{2m}^2(\infty) + \frac{(2m+3)(2m+2)(4m+5)}{(2m+1)(2m)(4m+1)}y_{2m+2}^{-2}(\infty)$$

and
$$C_2\int_{e^{-1}}^{\infty}V_2(x)\,dx = -y_2^0(\infty). \tag{4.6}$$

By definition $Y_{2m+1}(\tau) = 0$ which means that the odd C_n's satisfy the same equations in general as do the odd B_n's in ψ_0. There is a further equation for B_1 and B_3 which comes from matching the term in ρ^2 in ψ_1 when expressed in Oseen variables with the term in ρ^2 in $F_1(R)\Psi_1$. This gives

$$C_1\int_{e^{-1}}^{\infty}(5x^2-1)V_1(x)\,dx - \tfrac{2}{7}C_3v_1^2(\infty) = \tfrac{5}{48}e^2B. \tag{4.7}$$

It is then found that the only solution for the odd C's is

$$C_1 = \frac{eB^2}{24} \quad \text{and} \quad C_{2m+1} = (2m+2)(2m+1)(4m+3)\frac{C_1}{6}. \tag{4.8}$$

The equation (4.5) then becomes

$$\psi_1 = \frac{B}{48}\left\{(r^2-e^2) - \frac{Br}{6} + \frac{B(1+e^2)}{6e} V_1(r/e)\right\}(1-\mu^2) + \sum_{m=1}^{\infty} X_{2m}(r,e)\, U_{2m}(\mu), \tag{4.9}$$

where $X_{2m}(r,e)$ stands for the coefficient of $U_{2m}(\mu)$ in (4.5).

5. The third Stokes terms

Proudman & Pearson in their treatment of the sphere case show that the assumption $f_2(R) = R^2$ leads to a particular integral for ψ_2 containing a term in $r^2 \log r U_1(\mu)$. They infer from this that the assumption about $f_2(R)$ should be amended to

$$f_2(R) = R^2 \log R \tag{5.1}$$

and that ψ_2 is then a multiple of ψ_0. It is easy to show by differentiation that if the particular integral for ψ_2 contains a term in $r^2 \log r U_1(\mu)$ then this term arises from a term in $r^{-2}U_1(\mu)(1-\mu^2)^{-1}$ on the right-hand side of the inhomogeneous equation for l_2 and only from a term of this form there. This suggests that in the present case it is necessary to expand the right-hand side of

$$\bar{E}_r^2 l_2 = \frac{\partial(\psi_0, l_1)}{\partial(r,\mu)} + \frac{\partial(\psi_1, l_0)}{\partial(r,\mu)} \tag{5.2}$$

(which equation arises if $f_2(R) = R^2$) in powers of τ and to select the term in $\tau^{-2}U_1(\mu)(1-\mu^2)^{-1}$, a knowledge of whose numerical coefficient enables one to determine what multiple of ψ_0 the term ψ_2 must be.

On performing the expansion it is found that the right-hand side of (5.2) contains the term

$$\frac{-B^2}{120e^2\tau^2}\frac{U_1(\mu)}{(1-\mu^2)} \tag{5.3}$$

and therefore, if $f_2(R) = R^2$, the general solution for ψ_2 will be of the form

$$\left\{D_1 U_1(\tau) - \frac{e^2B^2}{720}\tau^2 \log \tau\right\}(1-\mu^2) + \left\{\begin{array}{l}\text{other functions of } \mu \text{ and } \tau \text{ not involving} \\ \text{the combination } \tau^2 \log \tau(1-\mu^2)\end{array}\right\}, \tag{5.4}$$

where D_1 is a constant.

But when this is expressed in Oseen variables it will contribute a term of order $R^2 \log R$ to Ψ and there are no such terms in the expansion of Ψ for small ρ. If there were such terms there would be terms of order $R \log R$ in L and these terms could not be matched since there are no terms of order $R \log R$ in l. Therefore

$$D_1 = -\frac{e^2B^2}{720}\log R + O(1) \tag{5.5}$$

and it must be assumed not that $f_2(R) = R^2$ but that (5.1) holds. ψ_2 is then a multiple of ψ_0 and by (5.5)

$$\psi_2 = \frac{B^2}{720}\left\{(r^2-e^2) - \frac{Br}{6} + \frac{B(1+e^2)}{6} V_1(r/e)\right\}(1-\mu^2). \tag{5.6}$$

6. Oblate spheroids

The equations of motion for an oblate spheroid can be derived from those for the prolate case by replacing r with ir. The appropriate general solutions then involve $U_n(ir)$ and $V_n(ir)$ which, from considerations of parity, are either real or purely imaginary. The boundary conditions on the surface of an oblate spheroid will be slightly different from those in the prolate case if the original definition of the Reynolds number is to be maintained throughout. The major axis of the axial section of an oblate spheroid is normal to the axis of symmetry whereas in the prolate case it is coincident with it. The member of the confocal family whose semi-major axis is unity is specified by $r = \sqrt{(1-e^2)}$ and consequently the boundary conditions on the body become

$$\psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{on} \quad r = \sqrt{(1-e^2)}. \tag{6.1}$$

e	B (prolate)	b (oblate)
0·0	9·00	9·00
0·1	8·96	8·99
0·2	8·85	8·96
0·3	8·67	8·93
0·4	8·40	8·85
0·5	8·04	8·76
0·6	7·56	8·64
0·7	6·95	8·50
0·8	6·13	8·32
0·9	4·96	8·09
1·0	0·00	7·64

Table 1. Values of B and b.

With these modifications the analysis can proceed as before to give

$$\psi_0 = \frac{1}{2}\left\{(r^2+e^2) - \frac{br}{6} + \frac{ib}{6e} V_1(ir/e)\right\}(1-\mu^2), \tag{6.2}$$

where
$$b = 12e^3\left\{e\sqrt{(1-e^2)} + (2e^2-1)\arctan\frac{e}{\sqrt{(1-e^2)}}\right\}^{-1}. \tag{6.3}$$

When $e \to 0$, $b \to 9$ and the Stokes result is recovered as before.

One can proceed to higher approximations as in the prolate case the only essential difference in the analysis being the value of the constant b and the change in the lower limits of the particular integrals in the Stokes solutions due to the change in boundary conditions on the surface.

It is of some interest to calculate the constants B and b, for these give a rough indication of the degree of departure of the flow from that about a sphere taken as a norm. Values of B and b are given in table 1.

7. The drag coefficient

By observing that ψ_2 is a multiple of ψ_0 and that ψ_1 consists of a multiple of ψ_0 plus a function which is odd in μ and which therefore makes no contribution to the drag, the calculation of the drag coefficient C_0 is greatly simplified. It is found that for the prolate case

$$C_0 = \frac{2\pi B}{(1-e^2)R}\left\{1+\frac{BR}{24}+\frac{B^2}{360}R^2\log R+O(R^2)\right\} \tag{7.1}$$

and for the oblate case

$$C_0 = \frac{2\pi b}{3R}\left\{1+\frac{bR}{24}+\frac{b^2}{360}R^2\log R+O(R^2)\right\}. \tag{7.2}$$

The first two terms in each of these were given by Oseen (1927).

The oblate case $e = 1$ corresponds to a circular disk broadside on to the stream. If $e \to 1$ in (7.2) then the drag coefficient for this disk is

$$C_0 = \frac{16}{R}\left\{1+\frac{R}{\pi}+\frac{8R^2}{5\pi^2}\log R+O(R^2)\right\}. \tag{7.3}$$

For prolate bodies the case $e = 1$ corresponds to a needle of length two units lying along the axis of symmetry. From (7.1) the drag on this needle is predicted to be zero. This is a rather surprising result for it implies that although the fluid is brought to rest along a finite line segment by the imposition of the boundary conditions this has no effect on the uniform stream as a whole. Looked at in another way this means that any number of these needles could be placed in and parallel to a uniform stream without disturbing it. Of course, this needle is a highly idealized case, being an entity of finite length but no breadth. Further, for a body whose greatest diameter is comparable with the distance between neighbouring molecules of the fluid, the original equations of motion can scarcely be expected to hold but should be replaced by equations taking into account the properties of the medium considered as a collection of particles rather than as a homogeneous fluid.

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